

Proof that $\frac{\bar{X} - \mu}{s/\sqrt{n}}$ is T-distributed

First, note that

$$\frac{n-1}{\sigma^2} s^2 \sim \chi_{(n-1)}^2,$$

and that

$$\frac{Z}{\sqrt{\chi_{(n-1)}^2/(n-1)}} \sim t_{(n-1)}.$$

Then,

$$\begin{aligned} \frac{\bar{X} - \mu}{s/\sqrt{n}} &= \frac{(\bar{X} - \mu)/\sigma}{s/(\sqrt{n}\sigma)} && \text{(Multiplying and dividing by } \sigma) \\ &= \frac{(\bar{X} - \mu)/\frac{\sigma}{\sqrt{n}}}{s/\sigma} && \text{(Moving the } \sqrt{n} \text{ factor to the numerator)} \\ &= \frac{Z}{\sqrt{\frac{(n-1)s^2}{(n-1)\sigma^2}}} && \text{(Multiplying and dividing by } n-1) \\ &= \frac{Z}{\sqrt{\chi_{(n-1)}^2/(n-1)}} = t_{(n-1)}. \end{aligned}$$