HEART: Statistics and Data Science With Networks

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Outline

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Matrices

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Example:

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What is M_{12}?

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For a matrix M, we let M_{ij} denote the entry of M in the i'th row and j'th column.

Example:

$$M = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}$$

What is M_{12}? Look in the first row and second column – yields $M_{12} = 0$.

Matrix-Matrix Multiplication

For two matrices M_1 and M_2, where M_1 is a $p_1 \times p_2$ matrix and M_2 is a $p_2 \times p_3$ matrix, the product matrix $M_1 M_2$ satisfies

$$(M_1 M_2)_{ij} = \sum_{k=1}^{p_2} (M_1)_{ik} (M_2)_{kj}.$$

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so the inner dimensions match, and the new matrix dimensions are the two outer dimensions.

Matrix-Matrix Multiplication: Example 1

Suppose

$$M_1 = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}; \qquad M_2 = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

What is $M_1 M_2$?

Matrix-Matrix Multiplication: Example 1

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What is $M_1 M_2$?

$$M_1 M_2 = \begin{pmatrix} 1 \times 1 + 2 \times 3 & 1 \times 3 + 2 \times 1 \\ 1 \times 1 + 4 \times 3 & 1 \times 3 + 4 \times 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 13 & 7 \end{pmatrix}$$

Matrix-Matrix Multiplication: Example 2

Suppose

$$M_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix}; \qquad M_2 = \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \end{pmatrix}$$

What is $M_1 M_2$?

Matrix-Matrix Multiplication: Example 2

Suppose

$$M_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix}; \qquad M_2 = \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \end{pmatrix}$$

What is $M_1 M_2$?

$$M_1 M_2 = \begin{pmatrix} 1 \times 1 + 2 \times 3 + 3 \times 0 & 1 \times 3 + 2 \times 1 + 0 \times 3 \\ 1 \times 1 + 4 \times 3 + 5 \times 0 & 1 \times 3 + 4 \times 1 + 5 \times 0 \end{pmatrix}$$

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Matrix-Matrix Multiplication: Example 3

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Matrix-Matrix Multiplication: Example 3

Suppose

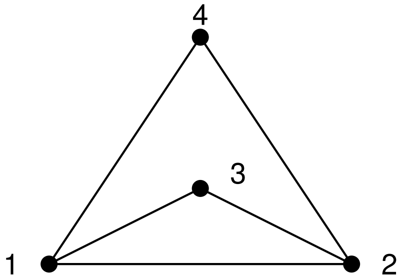
$$M_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix}; \qquad M_2 = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

What is $M_1 M_2$? Trick question! The inner dimensions don't match!

Matrix-Matrix Multiplication: Adjacency Matrix Multiplication

Recall the adjacency matrix of a graph:

Graph:



Adjacency Matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

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If A is the adjacency matrix of a graph, what is A^2?

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$$A^2 = AA = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

Matrix-Matrix Multiplication: Adjacency Matrix Multiplication

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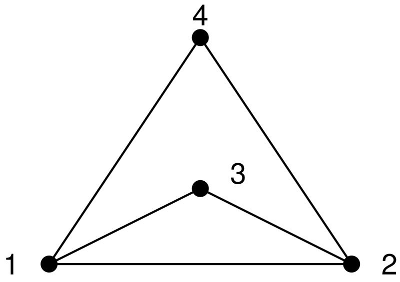

Adjacency Matrix squared:

$$\begin{pmatrix} 3 & 2 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

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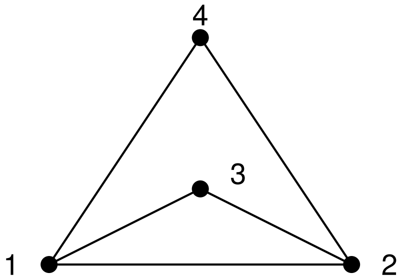

Adjacency Matrix squared:

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Observation: $(A^2)_{ij}$ counts the number of paths of length two from vertex i to vertex j!

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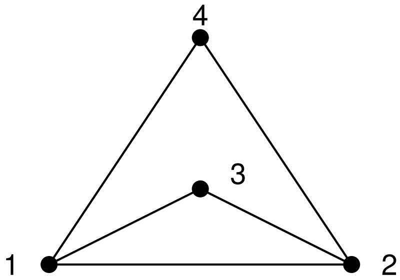

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$$(Mx)_{ij} = \sum_{k=1}^{p_2} M_{ik} x_{kj}.$$

But the second dimension is one, so $j = 1$. Therefore if M is a $p_1 \times p_2$ matrix, then Mx is a p_1-dimensional vector.

Matrix Vector Multiplication: Example 1

$$M = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}; \qquad x = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

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What is Mx?

$$Mx = \begin{pmatrix} 1 \times 2 + 1 \times 5 \\ 3 \times 2 + 3 \times 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 21 \end{pmatrix}.$$

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Let A be the adjacency matrix of a graph on n vertices, and let x be the vector of all ones. What is Ax?

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But what is $\sum_{k=1}^{n} A_{ik}$? It is the *degree* of the i'th vertex!

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1. Matrix-Matrix Multiplication
2. Matrix-Vector Multiplication
3. Eigenvalues

Eigenvalues

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Eigenvalues are very important quantities, but are hard to understand without more detail. Hard to compute in general. Might be complex numbers! But *symmetric* matrices always have real-valued eigenvalues (spectral theorem).

Eigenvalue: Example

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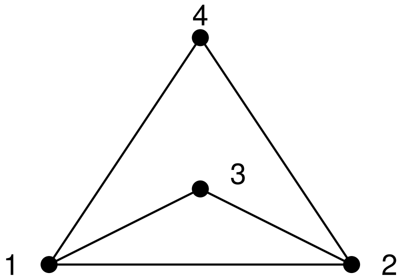
One eigenvalue $\lambda = 2$ with eigenvector $x = (1, 1)$. Another eigenvalue with $\lambda = 0$ with eigenvector $x = (1, -1)$.

Eigenvalue: Example

Recall the *combinatorial Laplacian* $D - A$, where D_{ii} is the degree of the i'th vertex.

Graph:



Combinatorial Laplacian Matrix:

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

Eigenvalue: Example

If we sum up the rows, this is the same as multiplying by a vector of all ones:

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- Zero is always an eigenvalue of the combinatorial Laplacian.
- Suppose we count the number of *different* eigenvectors with the eigenvalue zero. We call this the *eigenvalue multiplicity*.
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- Then the multiplicity of the eigenvalue zero is equal to the number of connected components of the graph.
- Shows how eigenvalues and eigenvectors of graph-related matrices provide valuable information of the graph!

Orthonormal Vectors

- Say a vector $x \in \mathbb{R}^n$ is a *unit vector* if $\|x\| = 1$, where

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

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- Say a collection of vectors $x_1 \cdots x_n$ are an orthonormal set if
  - $\|x_i\| = 1$ for all i
  - $\langle x_i, x_j \rangle = 0$ if $i \neq j$.

Orthonormal Vectors

Example:

$$x_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{1 in the } i\text{'th place}$$

Orthonormal Vectors

Example:

$$x_1 = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{pmatrix} \qquad x_2 = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \\ \frac{-1}{\sqrt{n}} \\ \vdots \\ \frac{-1}{\sqrt{n}} \end{pmatrix}$$

for n even.

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- Then M has n *real* eigenvalues $\lambda_1 \cdots \lambda_n$.
- M also has n orthonormal eigenvectors